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Parameter ranges for the existence of solutions whose state components have specified nodal structure in coupled multiparameter systems of nonlinear Sturm–Liouville boundary value problems

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Synopsis

The set of solutions to the two-parameter system

$$\left.\begin{aligned} -(p_1(x)u')' + q_1(x)u &= \lambda u + f(u, v)u \\ -(p_2(x)v')' + q_2(x)v &= \mu v + g(u, v)v \end{aligned}\right\} \text{ in } (a, b),$$

$$u(a) = u(b) = 0 = v(a) = v(b),$$

has been shown in a preceding paper of the author to exhibit a topological-functional analytic structure analogous to the structure of solution sets for nonlinear Sturm–Liouville boundary value problems. As the parameter λ and μ are varied, transitions in the solution set occur, first from trivial solutions to solutions $(u, 0)$ with u having n nodes on (a, b) or solutions $(0, v)$ with v having m nodes on (a, b), and then to solutions of the form (u, v), where u has n nodes on (a, b) and v has m nodes on (a, b), with n possibly different from m. Moreover, each transition is global in an appropriate bifurcation theoretic sense, with preservation of nodal structure. This paper explores these phenomena more closely, focusing on the range of parameters (λ, μ) for the existence of solutions (u, v) with u having n nodes on (a, b) and v having m nodes on (a, b) and its dependence on the assumptions placed on the coupling functions f and g. The principal tools of the analysis are the Alexander–Antman Bifurcation Theorem and *a priori* estimate techniques based on the maximum principle.

1. Introduction and examples

In a recent paper [**4**], the present author noted the existence of certain special types of continua of solutions to systems of boundary value problems of the form

$$\begin{aligned} L_1 u &\equiv -(p_1(x)u')' + q_1(x)u = \lambda u + f(u, v)u, \\ L_2 v &\equiv -(p_2(x)v')' + q_2(x)v = \mu v + g(u, v)v, \end{aligned} \tag{1.1}$$

where $x \in (a, b)$, and u and v are required to satisfy zero dirichlet boundary conditions. The functions p_i and q_i are assumed to be positive and continuously differentiable and positive and continuous on $[a, b]$, respectively, for $i = 1, 2$, while $f, g: \mathbb{R}^2 \to \mathbb{R}$ are assumed twice continuously differentiable with $f(0, 0) = 0 = g(0, 0)$. Then, under mild additional assumptions on f and g, if a pair (n, m) of *positive integers and a pair* (σ, τ) of sign orientations $(\sigma, \tau \in \{+, -\})$ are specified, there is a connected set $\mathscr{C}_{n,m,\sigma,\tau}$ of solutions to (1.1) in $\mathbb{R}^2 \times [C_0^1[a, b]]^2$ which is locally compact, of dimension $\geqq 2$ at every point, and such that if

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$(\lambda, \mu, u, v) \in \mathscr{C}_{n,m,\sigma,\tau}$, then u has $n-1$ simple zeros in (a, b), $sgnu'(a) = \sigma$, v has $m-1$ simple zeros in (a, b), and $sgnv'(a) = \tau$. The existence of such solution sets to (1.1) depends in a crucial way on the generalised Lotka–Volterra structure of (1.1). Witness that if (λ, u) solves the single equation

$$L_1 u = \lambda u + f(u, 0)u \tag{1.2}$$

on (a, b) with $u = 0$ at a and at b, then $(\lambda, \mu, u, 0)$ is a solution of (1.1) for any $\mu \in \mathbb{R}$. Consequently, to any continuum of dimension $\geqq 1$ of solutions to (1.2) in $\mathbb{R} \times C_0^1[a, b]$, there is a corresponding continuum of dimension $\geqq 2$ of solutions to (1.1) in $\mathbb{R}^2 \times [C_0^1[a, b]]^2$, where μ is "free" and $v \equiv 0$. Hence, by the Krasnosel'skii–Rabinowitz Bifurcation Theorem, there is a continuum $\mathscr{C}_{n,0,\sigma}$ of dimension $\geqq 2$ of solution to (1.1) with $v \equiv 0$, u having $n-1$ simple zeros in (a, b), and $sgnu'(a) = \sigma$ emanating in $\mathbb{R}^2 \times [C_0^1[a, b]]^2$ from $\mathbb{R}^2 \times \{(0, 0)\}$ along the line $\lambda = \lambda_n$, where λ_n is the unique eigenvalue of the linear problem

$$\begin{aligned} &L_1 w = \lambda w \quad \text{in } (a, b), \\ &w(a) = 0 = w(b), \end{aligned} \tag{1.3}$$

admitting an eigenfunction with $n-1$ simple zeros on (a, b) and $sgnw'(a) = \sigma$. Under appropriate technical assumptions on f, a set $\mathscr{C}_{n,m,\sigma,\tau}$ as described can be achieved as a bifurcation from $\mathscr{C}_{n,0,\sigma}$ by viewing $\mathscr{C}_{n,0,\sigma}$ as a set of "trivial" solutions and invoking the Alexander–Antman Multiparameter Bifurcation Theorem. Analogously, sets $\mathscr{C}_{n,m,\sigma,\tau}$ can also be obtained via secondary bifurcation from solution continua $\mathscr{C}_{0,m,\tau}$ with $u \equiv 0$, v having $m-1$ simple zeros on (a, b), and $sgnv'(a) = \tau$.

Some natural questions arise. Since there is more than one succession of primary and secondary bifurcations that lead to a set $\mathscr{C}_{n,m,\sigma,\tau}$, can there be more than one such set? If one of the parameters λ or μ is held fixed while the other is allowed to vary freely, is there a set $\mathscr{C}_{n,m,\sigma,\tau}$ that provides a link between solutions to (1.1) with $u \equiv 0$ and v having $m-1$ simple zeros in (a, b) with $sgnv'(a) = \tau$ and solutions to (1.1) with u having $n-1$ simple zeros in (a, b) with $sgnu'(a) = \sigma$ and $v \equiv 0$? What is the projection of $\mathscr{C}_{n,m,\sigma,\tau}$ into $\mathbb{R}^2$?

The purpose of this paper is to address these sorts of questions concerning the sets $\mathscr{C}_{n,m,\sigma,\tau}$. Some comment is in order at this point. Firstly, it is clear that the answers depend very much on the nature of f and g. Indeed in the case $n = m = 1$ and $\sigma = \tau = +$, u and v can be interpeted as the steady states to a population model for two interacting species which are allowed to move freely throughout a one-dimensional domain. Such problems and their analogues to higher dimensional space domains have been the focus of much recent activity in the differential equations and mathematical ecology communities. (See, for example, [**2, 5–10, 12–14, 16**] and the reference therein.) The nature of $\mathscr{C}_{1,1,+,+}$ in these articles depends on whether the model is mutualistic, competitive, or predatory. Consequently, there is no single all-encompassing answer to the particular questions we are asking about $\mathscr{C}_{n,m,\sigma,\tau}$. It is, however possible to identify some fairly general classes of f and g for which meaningful descriptions of $\mathscr{C}_{n,m,\sigma,\tau}$ can be given without becoming encyclopaedic. I began this process in [**4**, Section 3], noting additional assumptions on f and g (and L_1 and L_2) which were sufficient for $\mathscr{C}_{n,0,\sigma}$ and $\mathscr{C}_{0,m,\tau}$ to be linked together in $\mathbb{R}^2 \times [C_0^1[a, b]]^2$ by $\mathscr{C}_{n,m,\sigma,\tau}$ above a path in the parameter space $\mathbb{R}^2$ parallel to either the λ-axis or μ-axis. These assumptions on f and g are included in one of the classes treated in the subsequent sections of this paper. The extra assumptions on L_1 and L_2 in **[4]** are due to the use of the Sturm comparison theorem to get a necessary *a priori* estimate on the range of parameters for which (1.1) can have solutions (u, v) with u having $n-1$ simple zeros and v having $m-1$ simple zeros. The other classes of problems I want to discuss require a fuller, more flexible use of the power of the Alexander–Antman results **[1]**. One consequence of the more topological methods of this paper is that by taking different kinds of one-dimensional restrictions of parameter space $\mathbb{R}^2$, the extra assumptions on L_1 and L_2 in **[4**, Section 3] may be eliminated.

The subsequent sections of this article are organised as follows. In Section 2, classes of assumptions to place on f and g are identified. In Section 3 there follows the requisite *a priori* bounds on the $[C_0^1[a, b]]^2$ norm of solutions to (1.1). Finally, in Section 4, bifurcation theoretic arguments are used to address questions on $\mathscr{C}_{n,m,\sigma,\tau}$. However, before beginning the analysis, it will be very useful to consider a nontrivial situation where $\mathscr{C}_{n,m,\sigma,\tau}$ can be explicitly described. To this end, consider the special case of (1.1) given by

$$\begin{aligned} L_1 u &= \lambda u - \left(\int_0^1 Au^2\,dx + \int_0^1 Bv^2\,dx\right)u \quad \text{in } (0,1), \\ L_2 u &= \mu v - \left(\int_0^1 Cu^2\,dx + \int_0^1 Dv^2\,dx\right)v, \end{aligned} \tag{1.4}$$

with zero dirichlet boundary conditions on u and v, where A, B, C, D are constants with $A, D > 0$. Let $0 < \sigma_1 < \sigma_2 < \ldots, \sigma_n \to +\infty$ as $n \to \infty$, denote the sequence of eigenvalues to the linear problem

$$\begin{aligned} L_1 w &= \sigma w \quad \text{in } (0,1), \\ w(0) &= 0 = w(1), \end{aligned}$$

with corresponding eigenfunctions w_i, $i = 1, 2, \ldots,$ where w_i has $i-1$ simple zeros in $(0, 1)$, $w_i'(0) > 0$ and $\int_0^1 w_i^2\,dx = 1$. Likewise, let $0 < \gamma_1 < \gamma_2 < \ldots, \gamma_m \to \infty$ as $m \to \infty$, denote the sequence of eigenvalues to

$$\begin{aligned} L_2 y &= \gamma y \quad \text{in } (0,1), \\ y(0) &= 0 = y(1), \end{aligned}$$

with corresponding eigenfunctions y_j, $j = 1, 2, \ldots,$ where y_j has $j-1$ simple zeros in $(0, 1)$, $y_j'(0) > 0$, and $\int_0^1 y_j^2\,dx = 1$. Then, if $u \neq 0$ satisfies the boundary value problem

$$\begin{aligned} L_1 u &= \lambda u - A\left(\int_0^1 u^2\,dx\right)u \quad \text{in } (0,1), \\ u(0) &= 0 = u(1), \end{aligned} \tag{1.5}$$

it is straightforward to see that $\lambda - A\int_0^1 u^2\,dx = \sigma_i$ for some $i \geqq 1$ and $u = sw_i$ for some $s \in \mathbb{R}$, $s \neq 0$. It follows that the nontrivial solutions to (1.5) are described by

$$\left\{\left(\lambda, \pm\sqrt{\frac{\lambda-\sigma_i}{A}}\, w_i\right) : \lambda > \sigma_i\right\}$$

for $i = 1, 2, \ldots$. Similarly, the nontrivial solutions to

$$L_2 v = \mu v - D\left(\int_0^1 v^2\,dx\right)v \quad \text{in } (0,1), \qquad v(0) = 0 = v(1), \tag{1.6}$$

are given by

$$\left\{\left(\mu, \pm\sqrt{\frac{\mu-\gamma_j}{D}}\, y_j\right) : \mu > \gamma_j\right\}$$

for $j = 1, 2, \ldots$. Consequently, (1.4) has solution continua $\mathscr{C}_{n,0,\pm}$ and $\mathscr{C}_{0,m,\pm}$ given by

$$\left\{\left(\lambda, \mu, \pm\sqrt{\frac{\lambda-\sigma_n}{A}}\, w_n, 0\right) : \lambda \geqq \sigma_n,\ \mu \in \mathbb{R}\right\}$$

and

$$\left\{\left(\lambda, \mu, 0, \pm\sqrt{\frac{\mu-\gamma_m}{D}}\, y_m\right) : \lambda \in \mathbb{R},\ \mu \geqq \gamma_m\right\},$$

respectively. From [**4**, Theorem 2.1], if $u = \pm\sqrt{\frac{\lambda-\sigma_n}{A}}\, w_n$ for some $\lambda > \sigma_n$, a bifurcation from $\mathscr{C}_{n,0,\pm}$ to $\mathscr{C}_{n,m,\pm,\pm}$ occurs when

$$\left(L_2 + C\int_0^1 u^2\,dx\right)v = \mu v \quad \text{in } (0,1), \qquad v(0) = 0 = v(1), \tag{1.7}$$

has an eigenfunction with $m-1$ simple zeros. Since $C\int_0^1 u^2\,dx = (C/A)(\lambda - \sigma_n)$, (1.7) reduces to

$$L_2 v = (\mu + (C/A)(\sigma_n - \lambda))v \quad \text{in } (0,1), \qquad v(0) = 0 = v(1).$$

Hence, the transition occurs along the line segment

$$\mu + (C/A)(\sigma_n - \lambda) = \gamma_m \tag{1.8}$$

where $\lambda \geqslant \sigma_n$. Analogously, a transition from $\mathscr{C}_{0,m,\pm}$ to $\mathscr{C}_{n,m,\pm,\pm}$ occurs along the line segment

$$\lambda + (B/D)(\gamma_m - \mu) = \sigma_n, \tag{1.9}$$

$\mu \geqq \gamma_m$. Note that (1.8) and (1.9) can be expressed as

$$\mu = (C/A)\lambda + \gamma_m - (C/A)\sigma_n, \quad \lambda \geqq \sigma_n, \tag{1.10}$$

$$\mu = (D/B)\lambda + \gamma_m - (D/B)\sigma_n, \quad \mu \geqq \gamma_m. \tag{1.11}$$

Let us now describe $\mathscr{C}_{n,m,\pm,\pm}$. So far, there are no assumptions on the constants A, B, C, D other than positivity for A and D. I want to consider several possibilities. First, assume that C, B, and $AD-BC$ are positive. Solving (1.4) requires that

$$\lambda - \left(\int_0^1 (Au^2+Bv^2)\,dx\right) = \sigma_n, \quad u = sw_n,$$

$$\mu - \left(\int_0^1 (Cu^2+Dv^2)\,dx\right) = \gamma_m, \quad v = ty_m.$$

Hence $\lambda \geqq \sigma_n$, $\mu \geqq \gamma_m$, and

$$\begin{aligned} As^2+Bt^2 &= \lambda-\sigma_n, \\ Cs^2+Dt^2 &= \mu-\gamma_m. \end{aligned} \tag{1.12}$$

Since $AD-BC \neq 0$, (1.12) yields

$$s^2 = \frac{D(\lambda-\sigma_n)-B(\mu-\gamma_m)}{AD-BC}$$

$$t^2 = \frac{A(\mu-\gamma_m)-C(\lambda-\sigma_n)}{AD-BC}.$$

Since $AD-BC>0$, a solution to (1.4) with u having $n-1$ simple zeros in $(0,1)$ and v having $m-1$ simple zeros in $(0,1)$ is possible only when

$$\begin{aligned} D(\lambda-\sigma_n)-B(\mu-\gamma_m) &> 0, \\ A(\mu-\gamma_m)-C(\lambda-\sigma_n) &> 0. \end{aligned} \tag{1.13}$$

Of course, (1.13) reduces to

$$(C/A)\lambda+\gamma_m-(C/A)\sigma_n < \mu < (D/B)\lambda+\gamma_m-(D/B)\sigma_n, \tag{1.14}$$

$\lambda>\sigma_n$. As a consequence, $\mathscr{C}_{n,m,\pm,\pm}$ is given by

$$\left\{\left(\lambda,\mu,\pm\sqrt{\frac{D(\lambda-\sigma_n)-B(\mu-\gamma_m)}{AD-BC}}\,w_n,\ \pm\sqrt{\frac{A(\mu-\gamma_m)-C(\lambda-\sigma_n)}{AD-BC}}\,y_m\right):\right.$$
$$\left.(\lambda,\mu)\text{ satisfies (1.14)},\ \lambda>\sigma_n\right\}. \tag{1.15}$$

Hence $\mathscr{C}_{n,m,\pm,\pm}$ forms a two-dimensional sheet that meets $\mathscr{C}_{n,0,\pm}$ along the line segment (1.10) and $\mathscr{C}_{0,m,\pm}$ along (1.11). (See Figure 1.1.) If now, C and B are positive while $AD-BC<0$, the order of the inequalities in (1.14) is reversed. Consequently, $\mathscr{C}_{n,m,\pm,\pm}$ emanates from $\mathscr{C}_{n,0,\pm}$ via parameter values lying below the ray (1.10) as opposed to the case $AD-BC>0$, where $\mathscr{C}_{n,m,\pm,\pm}$ emanates from $\mathscr{C}_{n,0,\pm}$ via parameter values lying above the ray (1.10). Likewise, $\mathscr{C}_{n,m,\pm,\pm}$ emanates from $\mathscr{C}_{0,m,\pm}$ via parameter values to the left of ray (1.11), as opposed to the case $AD-BC>0$, where $\mathscr{C}_{n,m,\pm,\pm}$ emanates from $\mathscr{C}_{0,m,\pm}$ via parameter values lying to the right of ray (1.11). (See Figure 1.2.) Otherwise, the case when $C, B>0$ and $AD-BC<0$ is very similar to the case when $C, B>0$ and $AD-BC>0$. However, if $C, B>0$ and $AD-BC=0$, a noteworthy difference

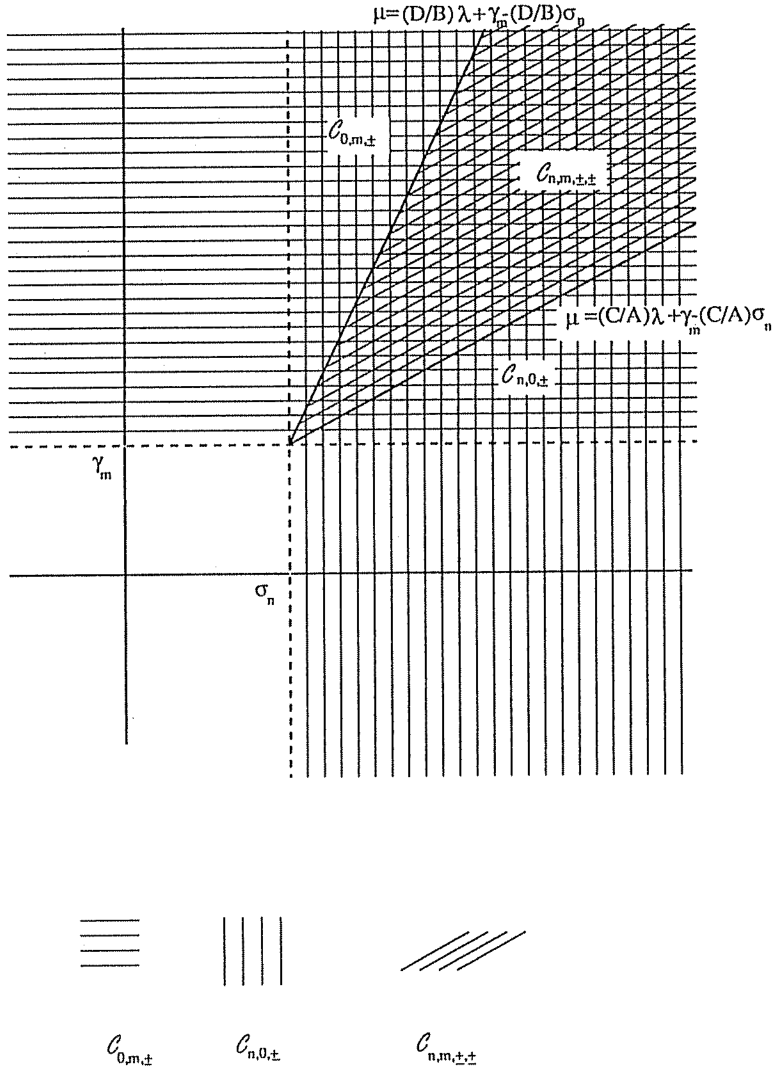

Figure 1.1. Projections of the solution continua $\mathscr{C}_{0,m,\pm}$, $\mathscr{C}_{n,0,\pm}$, $\mathscr{C}_{n,m,\pm,\pm}$ for (1.4) into $\lambda-\mu$ parameter space in the case (A, B, C, D, $AD-BC>0$).

occurs. It is easy to see, by multiplying the first equation of (1.12) by D and the second by B and then subtracting, that solutions to (1.4) with u having $n-1$ simple zeros and v having $m-1$ simple zeros occur only when λ and μ are constrained to lie on the ray $\mu=(D/B)\lambda+\gamma_m-(D/B)\sigma_n$, $\lambda \geqq \sigma_n$. In this case, $\mathscr{C}_{n,m,\pm,\pm}$ is a two-dimensional sheet above this ray, and the transitions to $\mathscr{C}_{n,0,\pm}$ or $\mathscr{C}_{0,m,\pm}$ are, in the language of bifurcation diagrams, "vertical". I note that such a phenonmenon has been observed under certain conditions by Cosner and Lazer [7] and Blat and Brown [2] in the context of the steady-state equations for a competitive Lotka–Volterra model with diffusion.

In any of the cases of (1.4) considered so far, $C>0$ and $B>0$, and the

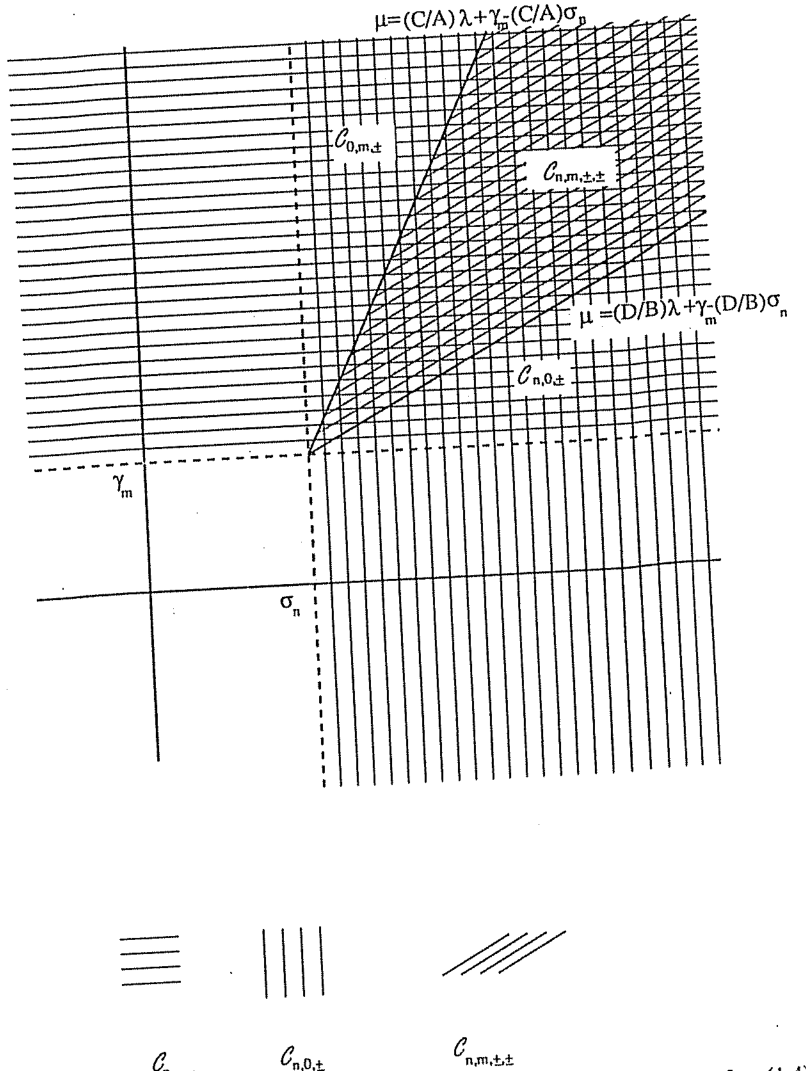

Figure 1.2. Projections of the solution continua $\mathscr{C}_{0,m,\pm}$, $\mathscr{C}_{n,0,+}$, $\mathscr{C}_{n,m,\pm,\pm}$ for (1.4) into $\lambda-\mu$ parameter space in the case $(A, B, C, D>0, AD-BC<0)$.

projection of $\mathscr{C}_{n,0,\pm,\pm}$ into $\mathbb{R}^2$ is contained in $\{(\lambda, \mu): \lambda>\sigma_n, \mu>\gamma_m\}$, the intersection of the projections into $\mathbb{R}^2$ of $\mathscr{C}_{n,0,\pm}$ and $\mathscr{C}_{0,m,\pm}$. If either $B<0$ or $C<0$, such is not the case. For instance, if $B<0$ while $C>0$, then necessarily $AD-BC>0$ and the existence of solutions to (1.4) with u having $n-1$ simple zeros in (0, 1) and v having $m-1$ simple zeros in (0, 1) is again equivalent to (1.13). However, since $B<0$, (1.13) in this case reduces to

$$\begin{cases}\mu>(D/B)\lambda+\gamma_m-(D/B)\sigma_n,\\ \mu>(C/A)\lambda+\gamma_m-(C/A)\sigma_n,\end{cases} \tag{1.16}$$

and $\mathscr{C}_{n,m,\pm,\pm}$ is given by

$$\left\{\left(\lambda,\ \mu,\ \pm\sqrt{\frac{D(\lambda-\sigma_n)-B(\mu-\gamma_m)}{AD-BC}}\,w_n,\ \pm\sqrt{\frac{A(\mu-\gamma_m)-C(\lambda-\sigma_n)}{AD-BC}}\,y_m\right):(\lambda,\mu)\text{ satisfies (1.16)}\right\}. \quad (1.17)$$

(See Figure 1.3.) If now $B<0$ and $C<0$, then $AD-BC$ can be positive, negative, or zero. If $AD-BC>0$, the projection of $\mathscr{C}_{n,m,\pm,\pm}$ into $\mathbb{R}^2$ is again given by (1.13), which in this case yields (1.16). (See Figure 1.4.) If now $AD-BC<0$, the inequalities in (1.13) must be reversed and the projection of

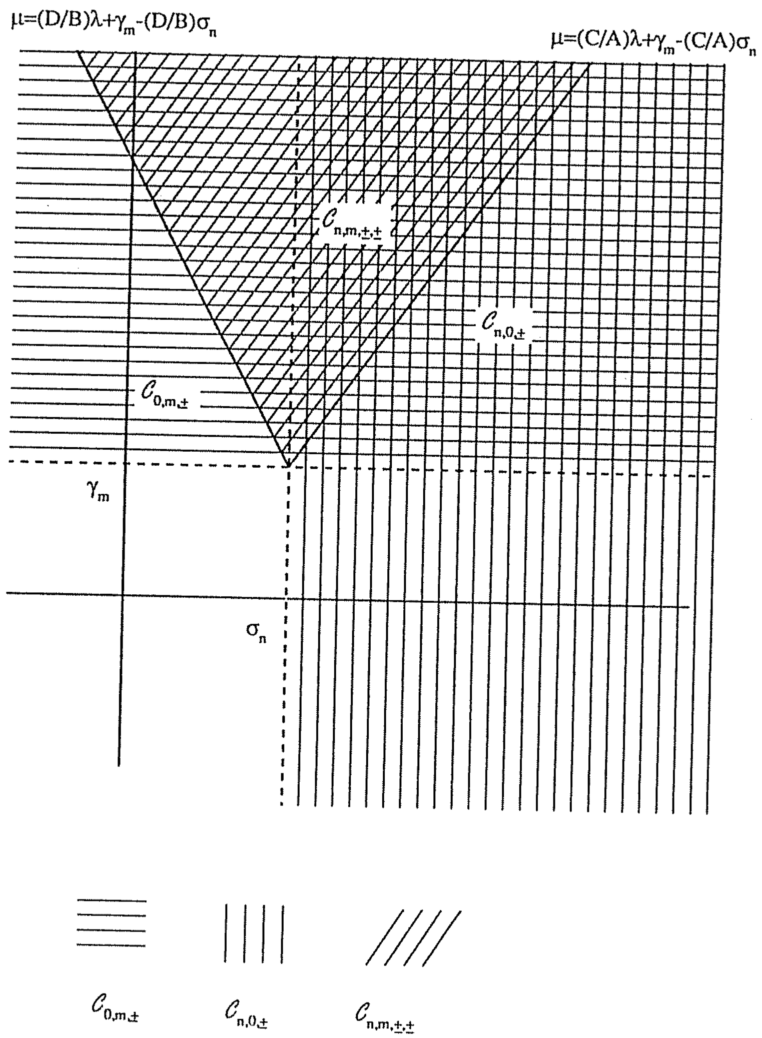

Figure 1.3. Projections of the solution continua $\mathscr{C}_{0,m,\pm}$, $\mathscr{C}_{n,0,\pm}$, $\mathscr{C}_{n,m,\pm,\pm}$ for (1.4) into $\lambda-\mu$ parameter space in the case $(A, C, D>0,\ B<0)$.

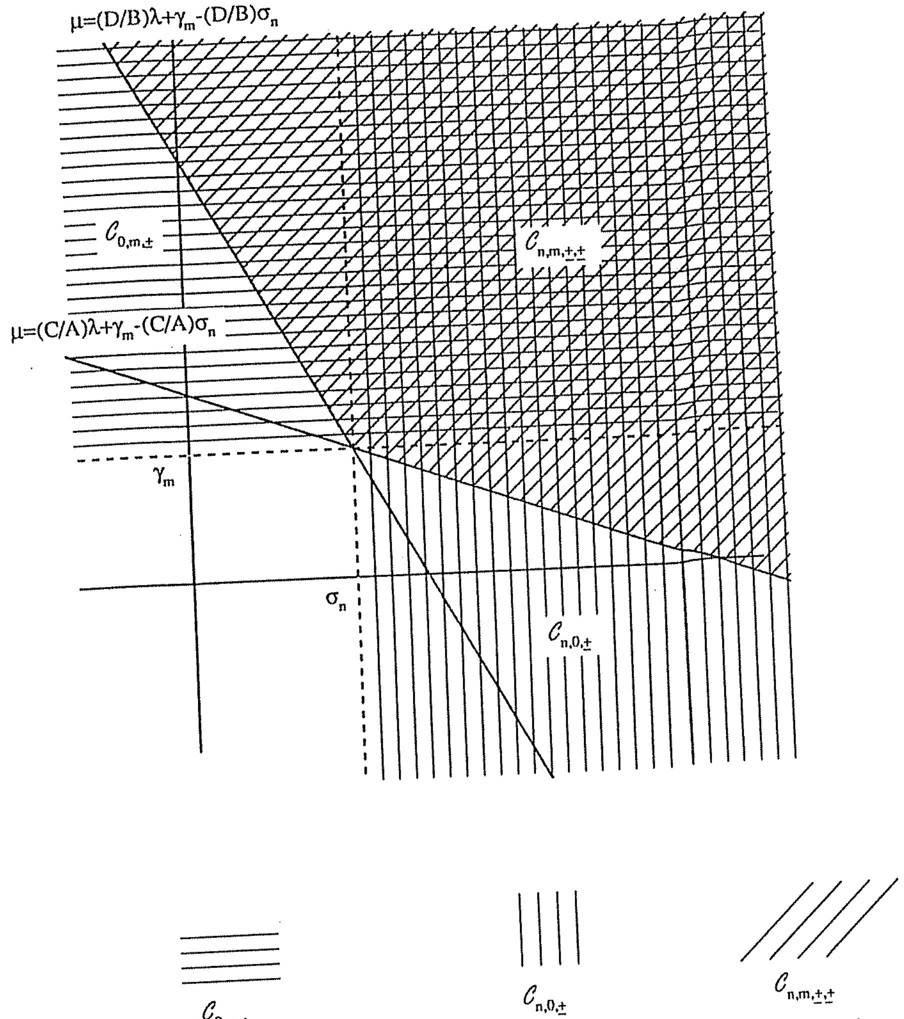

Figure 1.4. Projections of the solution continua $\mathscr{C}_{0,m0\pm}$, $\mathscr{C}_{n,0,+}$, $\mathscr{C}_{n,m,\pm,\pm}$ for (1.4) into $\lambda-\mu$ parameter space in the case $(A,\ D>0,\ B,\ C<0,\ AD-BC>0)$.

$\mathscr{C}_{n,m,\pm,\pm}$ into $\mathbb{R}^2$ is given by

$$\begin{cases}\mu<(D/B)\lambda+\gamma_m-(D/B)\sigma_n,\\ \mu<(C/A)\lambda+\gamma_m-(C/A)\sigma_n.\end{cases} \tag{1.18}$$

It is immediate from (1.18) that the projection of $\mathscr{C}_{n,m,\pm,\pm}$ into $\mathbb{R}^2$ does not intersect $\{(\lambda,\mu)\colon \lambda>\sigma_n,\ \mu>\gamma_m\}$. (See Figure 1.5.)

All of the cases of (1.4) discussed so far have the common feature that for any fixed (λ,μ) the set $\{(u,v)\colon (\lambda,\mu,u,v)\in\mathscr{C}_{n,m,\pm,\pm}\}$ is a bounded set in $[C_0^1[0,1]]^2$. If $A,\ D>0,\ B,\ C<0$, and $AD-BC=0$, such is no longer the case. Witness that $\mathscr{C}_{n,0,\pm}$ must meet $\mathscr{C}_{n,m,\pm,\pm}$ when the parameters (λ,μ) lie along the ray $\mu=(C/A)\lambda+\gamma_m-(C/A)\sigma_n\equiv(D/B)\lambda+\gamma_m-(D/B)\sigma_n$, $\lambda\geqq\sigma_n$, while $\mathscr{C}_{0,m,\pm}$

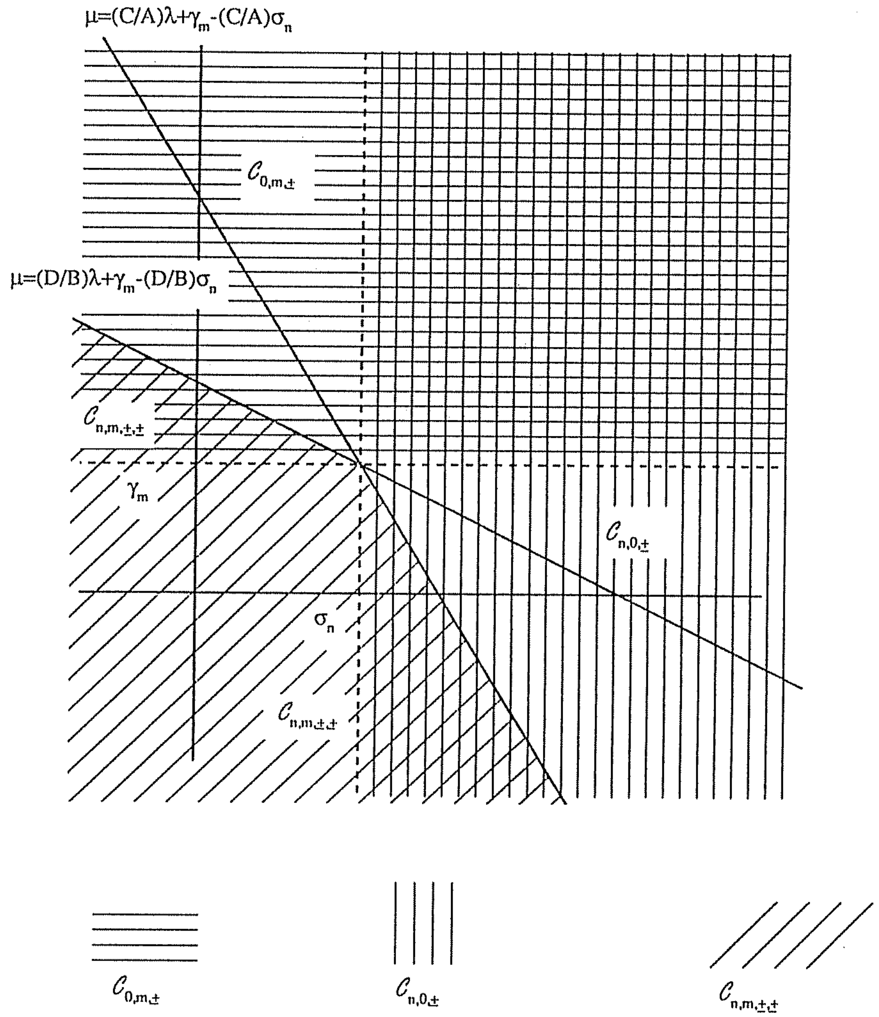

Figure 1.5. Projections of the solution continua $\mathscr{C}_{0,m,\pm}$, $\mathscr{C}_{n,0,\pm}$, $\mathscr{C}_{n,m,\pm,\pm}$ for (1.4) into $\lambda-\mu$ parameter space in the case $(A, D>0, B, C<0, AD-BC<0)$.

meets $\mathscr{C}_{n,m,\pm,\pm}$ when the parameters lie on the same line but with $\mu \geqq \gamma_m$. These rays have only the point (σ_n, γ_m) in common. Hence, if for example $\lambda_0 > \sigma_n$ is fixed and (λ, μ) is constrained to the line $\lambda = \lambda_0$, the transition from $\mathscr{C}_{n,0,\pm}$ to $\mathscr{C}_{n,m,\pm,\pm}$ occurs at the point $(\lambda_0, (D/B)\lambda_0 + \gamma_m - (D/B)\sigma_n)$ and moreover, the projection of $C_{n,m,\pm,\pm}$ into $\mathbb{R}^2$ meets the line $\lambda = \lambda_0$ only at the point $(\lambda_0, (D/B)\lambda_0 + \gamma_m - (D/B)\sigma_n)$. Hence, $\{(u, v): (\lambda_0, (D/B)\lambda_0 + \gamma_m - (D/B)\sigma_n, u, v) \in \mathscr{C}_{n,m,\pm,\pm}\}$ is unbounded in $[C_0^1[0, 1]]^2$. Additionally, if (λ, μ) is constrained to the line $\lambda = \lambda_0$, then $\mathscr{C}_{n,m,\pm,\pm}$ does not meet $C_{0,m,\pm}$. Analogous statements hold when $\mu_0 > \gamma_m$ is fixed.

Example (1.4) indicates a wide variety of possible answers to the questions raised earlier in this section. It is also valuable as it indicates the nature of the bifurcation phenomena for (1.1) in general, as will be observed in the subsequent

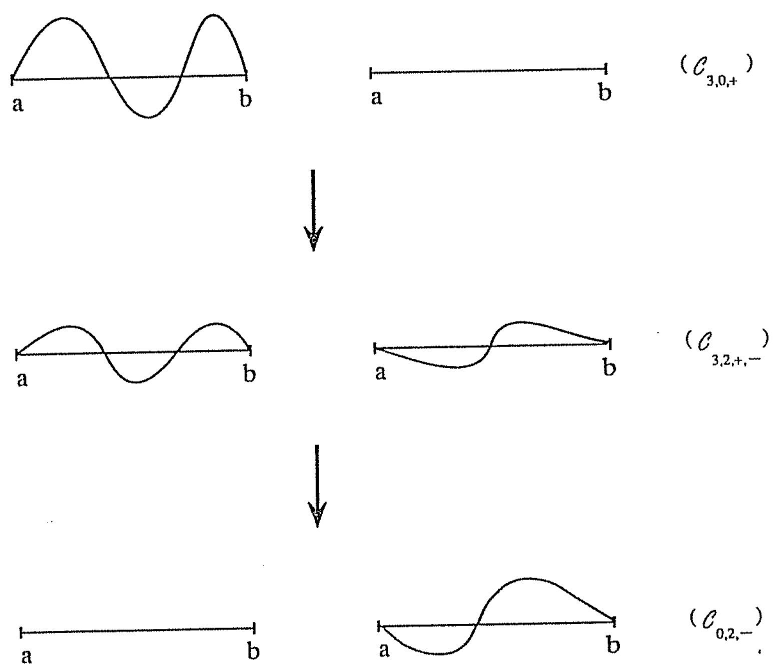

Figure 1.6. Solutions to (1.1) lying on $\mathscr{C}_{n,0,\tau}$, $\mathscr{C}_{n,m,\tau,\sigma}$, and $\mathscr{C}_{0,m,\sigma}$ with $n=3$, $m=2$, $\tau=+$, $\sigma=-$. The arrows indicate the bifurcations among the continua.

sections of this asrticle. Namely, there is a curve in the $\lambda-\mu$ plane along which a transition occurs (in $\mathbb{R}^2\times(C_0^1[a,b])^2$) from a solution continuum $\mathscr{C}_{n,0,\tau}$ (u components have n nodes, v components vanish identically) to a solution continuum $\mathscr{C}_{n,m,\tau,\sigma}$ (u components have n nodes, v components have m nodes). Moreover, there is a second curve in the $\lambda-\mu$ plane along which $\mathscr{C}_{n,m,\tau,\sigma}$ meets a solution continuum $\mathscr{C}_{0,m,\sigma}$ (u components vanish identically, v components have m nodes). (See Figure 1.6.) For (1.4), the first transition occurs along the infinite line segment (1.8) and the second along the infinite line segment (1.9). Two brief remarks are in order. Firstly, the location of (1.8) and (1.9) and the subsequent locus of $\mathscr{C}_{n,m,\tau,\sigma}$ depend on the assumptions placed on the nonlinearity in (1.4). Secondly, that (1.8) and (1.9) are infinite line segment in the case (1.4) is due to the "nonlocal" nature of the nonlinearity. For (1.1) in general, we cannot expect that the parametric curve for the transition from $\mathscr{C}_{n,0,\tau}$ to $\mathscr{C}_{n,m,\tau,\sigma}$ (or from $\mathscr{C}_{0,m,\tau}$ to $\mathscr{C}_{n,m,\tau,\sigma}$) is linear.

2. Classes of assumptions on f and g

The examples of the previous section suggest three interesting classes of assumptions to place on f and g. In the special case when u and v are positive and may be interpreted as population densities, the resulting problems may justifiably be classified as competitive, cooperative, and predatory. While these three classes of assumptions on f and g by no means exhaust all possibilities, I believe that the analysis of the resulting problems gives substantial insight into the nature of the solution structure to (1.1).

There are certain features common to all the problems considered in this article. To note them, let $0<\lambda_1<\lambda_2<\ldots, \lambda_n\to\infty$ as $n\to\infty$ and $0<\mu_1<\mu_2<\ldots, \mu_m\to\infty$ as $m\to\infty$ be the sequences of eigenvalues to the problems

$$L_1 w=\lambda w \quad \text{in } (a, b),$$
$$w(a)=0=w(b).$$

and

$$L_2 y=\mu y \quad \text{in } (a, b),$$
$$y(a)=0=y(b),$$

respectively. Then there are corresponding sequences $\{w_i\}_{i=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ of eigenfunctions with w_i having $i-1$ simple zeros in (a, b) and $w_i'(a)>0$ and y_j having $j-1$ simple zeros in (a, b) and $y_j'(a)>0$, for all $i, j\in Z^+$. Assume

(H1) (i) $f(x_1, 0)<0$, if $x_1\neq 0$.
(ii) $g(0, x_2)<0$, if $x_2\neq 0$.

Then, for all $n\in Z^+$, if

$$L_1 u=\lambda u+f(u, 0)u \quad \text{in } (a, b), \qquad u(a)=0=u(b), \tag{2.1}$$

and u has $n-1$ simple zeros in (a, b), then $\lambda>\lambda_n$. Likewise, for all $m\in Z^+$ if

$$L_2 v=\mu v+g(0, v)v \quad \text{in } (a, b), \qquad v(a)=0=v(b), \tag{2.2}$$

and v has $m-1$ simple zeros in (a, b), then $\mu>\mu_m$. Consequently, [**4**, Theorem 2.1] obtains and sets $\mathscr{C}_{n,m,\sigma,\tau}$ as described in Section 1 exist. For most of the remainder of this article, assume in addition

(H2) (i) For all $n\in Z^+$, if $\lambda>\lambda_n$, there are exactly two solutions to (2.1) having $n-1$ simple zeros in (a, b). These solutions lie along two unbounded continua $\mathscr{C}_n^+$ and $\mathscr{C}_n^-$ in $\mathbb{R}\times C_0^1[a, b]$ emanating from $\{(\lambda_n, 0)\}$. The solutions on $\mathscr{C}_n^+$ are characterised by having a positive derivative at a, while those on $\mathscr{C}_n^-$ have a negative derivative at a.

(ii) For all $m\in Z^+$, if $\mu>\mu_m$, there are exactly two solutions to (2.2) having $m-1$ simple zeros in (a, b). These solutions lie along two unbounded continua $\mathscr{C}_m^+$ and $\mathscr{C}_m^-$ in $\mathbb{R}\times C_0^1[a, b]$ emanating from $\{(\mu_m, 0)\}$. The solutions on $\mathscr{C}_m^+$ are characterised by having a positive derivative at a, while those on $\mathscr{C}_m^-$ have a negative derivative at a.

While assumption (H2) need not allways obtain, it is certainly not an unreasonably accompaniment to (H1). It is known to hold in a number of situations. It is also the simplest assumption to make concerning the solutions to (2.1) and (2.2), given (H1). Moreover, even when (H2) fails to hold, given (H1), it can usually be replaced with the weaker.

(H2)′ (i) For all $n\in Z^+$, there is $\delta_n>0$ so that for all $\lambda\in(\lambda_n, \lambda_n+\delta_n)$, there are exactly two solutions to (2.1) having $n-1$ simple zeros in (a, b), one whose derivative at a is positive and the other whose derivative at a is negative.

(ii) For all $m \in Z^+$, there is $\sigma_m > 0$ so that for all $\mu \in (\mu_m, \mu_m + \sigma_m)$, there are exactly two solutions to (2.2) having $m-1$ simple zeros in (a, b), one whose derivative at a is positive and the other whose derivative at a is negative.

Many of the results on $\mathscr{C}_{n,m,\sigma,\tau}$ still hold when (H2) is replaced by (H2)′, as will be seen at the end of the article.

At this point, the assumptions on f and g split into three classes: "competitive", "cooperative", and "predatory". In the "competitive" case, in addition to (H1) and (H2) (or (H2)′), assume

(H3)

$$\left.\begin{aligned} f(x_1, x_2) \leqq 0 \\ g(x_1, x_2) \leqq 0 \end{aligned}\right\} \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

(H4) There are continuous functions $h, k: \mathbb{R} \to [0, \infty]$ so that
(i) $|x_1| > h(\lambda)$ implies $\lambda + f(x_1, x_2) < 0$ for all $x_2 \in \mathbb{R}$;
(ii) $|x_2| > k(\mu)$ implies $\mu + g(x_1, x_2) < 0$ for all $x_1 \in \mathbb{R}$.

For the "cooperative" situation, in addition to (H1) and (H2), assume

(H5)

$$\left.\begin{aligned} f(0, x_2) > 0 \\ g(x_1, 0) > 0 \end{aligned}\right\} \quad \text{for all } x_1, x_2 \in \mathbb{R} - \{0\}.$$

(H6)

$$f(x_1, x_2) \leqq -p(x_1) + q(x_2),$$
$$g(x_1, x_2) \leqq r(x_1) - s(x_2),$$

where $p, q, r, s: \mathbb{R} \to [0, \infty)$ are continuous functions satisfying:
(i) $p(0) = q(0) = r(0) = s(0) = 0$;
(ii) $p(t), q(t), r(t), s(t) > 0$ if $t \neq 0$;
(iii) p and s are strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$;
(iv) There are $\alpha, \beta \in (0, 1)$ such that $r(t) \leqq \alpha p(t)$ and $q(t) \leqq \beta s(t)$ for all $t \in \mathbb{R}$.

The assumptions additional to (H1) and (H2) in the "predatory" case are a blend of those for the "competitive" and "cooperative" cases. In the "predatory" case, assume (H1), (H2), and

(H7)

$$f(x_1, x_2) \geqq 0 \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

(H8) There is a continuous function $j: \mathbb{R} \to [0, \infty)$ so that $|x_1| > j(\lambda)$ implies $\lambda + f(x_1, x_2) < 0$ for all $x_2 \in \mathbb{R}$.

(H9)

$$g(x_1, 0) > 0 \quad \text{for all } x_1 \in \mathbb{R} - \{0\}.$$

(H10)

$$g(x_1, x_2) \leqq m(x_1) - n(x_2),$$

where $m, n: \mathbb{R} \to [0, \infty)$ are continuous functions satisfying

(i) $m(0) = n(0) = 0$;

(ii) $m(t), n(t) > 0$ if $t \neq 0$;

(iii) n is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.

3. *A priori* bounds

In order to establish the *a priori* bounds on the $[C_0^1[a, b]]^2$ norm of solutions (u, v) that are needed in Section 4, it follows from the regularity theory of differential equations and the structure of (1.1) that it suffices to estalish *a priori* bounds on the $[C_0^0[a, b]]^2$ norm of solutions (u, v). The maximum principle is the main tool for this purpose. In the "competitive" case, it follows from (H4) as in [4] that if (u, v) solves (1.1) at (λ, μ), then $\|u\|_\infty \leqq h(\lambda)$ and $\|v\|_\infty \leqq k(\mu)$. The following result holds:

THEOREM 3.1. *Suppose that* (H1)–(H4) *are satisfied. Then if* (λ, μ, u, v) *satisfy* (1.1) *with u having $n-1$ simple zeros in (a, b) and v having $m-1$ simple zeros in (a, b), $\lambda > \lambda_n$ and $\mu > \mu_m$. Moreover, if W is any bounded subset of $\{(\lambda, \mu): \lambda > \lambda_n, \mu > \mu_m\}$, there is a $C(W) > 0$ such that if (λ, μ, u, v) solves* (1.1) *with u having $n-1$ simple zeros in (a, b), v having $m-1$ simple zeros in (a, b), and $(\lambda, \mu) \in W$, then*

$$\|(u, v)\|_{[C_0^1[a,b]]^2} \equiv \{\|u\|^2_{C_0^1[a,b]}\}^{\frac{1}{2}} < C(W).$$

In the "cooperative" case, the argument is more complicated. Suppose that (H1), (H2), (H5), and (H6) are met and that (λ, μ, u, v) solves (1.1) with u and v nonzero. Suppose further that $\lambda + \|q(v)\|_\infty > 0$ and that there is an $x_0 \in (a, b)$ with $u(x_0) > 0$ and $p(u(x_0)) > \lambda + \|q(v)\|_\infty$, where p, q are as in (H6) and $\|\ \|_\infty$ is the usual supremum norm. Then there is an open subinterval I of (a, b) such that $x_0 \in I$, $u(x) > 0$ and $p(u(x)) > \lambda + \|q(v)\|_\infty$ for $x \in I$, while $u(x) = \alpha^* = (p|_{[0,\infty]})^{-1}(\lambda + \|q(v)\|_\infty)$ on ∂I. From (1.1), it follows that on I

$$\begin{aligned} L_1(u - \alpha^*) &= (\lambda + f(u, v))u - q_1(x)\alpha^* \\ &\leqq (\lambda + q(v) - p(u))u - q_1(x)\alpha^* \\ &\leqq (\lambda + \|q(v)\|_\infty - p(u))u - q_1(x)\alpha^* \\ &\leqq 0, \end{aligned}$$

with $u - \alpha^* = 0$ on ∂I. The maximum principle implies that $u \leqq \alpha^*$ on I, a contradiction. Consequently if (λ, μ, u, v) solves (1.1) with $\lambda + \|q(v)\|_\infty > 0$, then for any $x \in (a, b)$ with $u(x) > 0$, $p(u(x)) \leqq \lambda + \|q(v)\|_\infty$. By repeating essentially the same argument, it follows that if (λ, μ, u, v) solves (1.1) with u and v nonzero, then $\lambda + \|q(v)\|_\infty > 0$ and $\|p(u)\|_\infty \leqq \lambda + \|q(v)\|_\infty$. Similarly, $\mu + \|r(u)\|_\infty > 0$ and $\|s(v)\|_\infty \leqq \mu + \|r(u)\|_\infty$. Hence

$$(\|p(u)\|_\infty - \|r(u)\|_\infty) + (\|s(v)\|_\infty - \|q(v)\|_\infty) \leqq \lambda + \mu. \tag{3.1}$$

Since $r(t) \leqq \alpha p(t)$ and $q(t) \leqq \beta s(t)$ for all $t \in \mathbb{R}$, where $\alpha, \beta \in (0, 1)$, (3.1) yields

$$(1 - \alpha)\ \|(p)u\|_\infty + (1 - \beta)\ \|s(v)\|_\infty \leqq \lambda + \mu. \tag{3.2}$$

As a result of (H6)(iii), *a priori* bounds on (u, v) are available as in the following theorem:

THEOREM 3.2. *Suppose that* (H1), (H2), (H5), *and* (H6) *are satisfied. Then if* (λ, μ, u, v) *satisfy* (1.1) *with* u *having* $n-1$ *simple zeros in* (a, b) *and* v *having* $m-1$ *simple zeros in* (a, b), $\lambda+\mu>0$. *Moreover, if* W *is any bounded subset of* $\{(\lambda, \mu): \lambda+\mu>0\}$, *there is a* $C(W)>0$ *such that if* (λ, μ, u, v) *solves* (1.1) *with* u *having* $n-1$ *simple zeros in* (a, b), v *having* $m-1$ *simple zeros in* (a, b) *and* $(\lambda, \mu)\in W$, *then* $\|(u, v)\|_{[C_0^1[a,b]]^2}<C(W)$.

Finally, consider the "predatory" case and suppose that (H1), (H2), and (H7)–(H10) hold. Let (λ, μ, u, v) solve (1.1) with u having $n-1$ simple zeros and v having $m-1$ simple zeros. Then from (H7) and (H8), $\lambda>\lambda_n$ and $\|u\|_\infty\leqq j(\lambda)$. Arguing as in the "cooperative" case, it follows from (H10) that $\mu+\|m(u)\|_\infty>0$ and $\|n(v)\|_\infty\leqq\mu+\|(m(u)\|_\infty$. As a consequence, if $\bar{m}(\lambda)=\max\{m(t): |t|\leqq j(\lambda)\}$, then $\mu>-\bar{m}(\lambda)$, $\|n(v)\|_\infty\leqq\mu+\bar{m}(\lambda)$, and the following theorem: is established:

THEOREM 3.3. *Suppose that* (H1), (H2), *and* (H7)–(H10) *are satisfied. Then if* (λ, μ, u, v) *satisfy* (1.1) *with* u *having* $n-1$ *simple zeros in* (a, b) *and* v *having* $m-1$ *simple zeros in* (a, b), $\lambda>\lambda_n$ *and* $\mu>-\bar{m}(\lambda)$, *where* $\bar{m}(\lambda)=\max\{m(t): |t|\leqq j(\lambda)\}$. *Moreover, if* W *is any bounded subset of* $\{(\lambda, \mu): \lambda>\lambda_n, \mu>-\bar{m}(\lambda)\}$, *there is a* $C(W)>0$ *such that if* (λ, μ, u, v) *solves* (1.1) *with* u *having* $n-1$ *simple zeros in* (a, b), v *having* $m-1$ *simple zeros in* (a, b), *and* $(\lambda, \mu)\in W$, *then* $\|(u, v)\|^2_{[C_0^1[a,b]]}<C(W)$.

4. Main result

Assume that (H1) and (H2) hold. For $n=1, 2, 3, \ldots$ and $\sigma=+, -$, let $U^{n,\sigma}(\lambda)$ be 0 if $\lambda\leqq\lambda_n$ and the unique element of $C_0^1[a, b]$ such that $(\lambda, U^{n,\sigma}(\lambda))\in\mathscr{C}_n^\sigma$ if $\lambda>\lambda_n$. Likewise, for $m=1, 2, 3, \ldots$, and $\tau=+, -$, $V^{m,\tau}(\mu)$ is 0 if $\mu\leqq\mu_m$ and such that $(\mu, V^{m,\tau}(\mu))\in\mathscr{C}_m^\tau$ if $\mu>\mu_m$. Then $\mathscr{C}_{n,0,\sigma}$ and $\mathscr{C}_{0,m,\tau}$ are given by

$$\{(\lambda, \mu, U^{n,\sigma}(\lambda), 0): \lambda\in\mathbb{R}, \mu\in\mathbb{R}\} \tag{4.1}$$

and

$$\{(\lambda, \mu, 0, V^{m,\tau}(\mu)): \lambda\in\mathbb{R}, \mu\in\mathbb{R}\}, \tag{4.2}$$

respectively. Next, for $n=1, 2, 3, \ldots$, $m=1, 2, 3, \ldots$ and $\sigma=+, -$, let $\mu_m^{n,\sigma}(\lambda)$ be the unique eigenvalue of

$$\begin{gathered} L_2 y-g(U^{n,\sigma}(\lambda), 0)y=\mu y \quad \text{in } (a, b), \\ y(a)=0=y(b), \end{gathered} \tag{4.3}$$

admitting an eigenfunction with $m-1$ simple zeros in (a, b). Then $\mu_m^{n,\sigma}(\lambda)$ is a continuous function of $\lambda\in\mathbb{R}$ and a transition from $\mathscr{C}_{n,0,\sigma}$ to a $\mathscr{C}_{n,m,\sigma,\pm}$ can occur only for parameter values along the curve $\mu=\mu_m^{n,\sigma}(\lambda)$, $\lambda\geqq\lambda_n$.

It is not too difficult to see that in fact such a transition does occur. From [**3**] and [**4**], (1.1) can be reformulated as

$$\vec{z}=N(\lambda, \mu, \vec{z})\equiv A(\lambda, \mu)\vec{z}+H(\lambda, \mu, \vec{z}), \tag{4.4}$$

where $\vec{z} = (z_1, z_2)' \in [C_0^1[a, b]]^2$, $A(\lambda, \mu): [C_0^1[a, b]]^2 \to [C_0^1[a, b]]^2$ is the compact linear operator given by

$$A(\lambda, \mu) \equiv \begin{pmatrix} L_1^{-1}\left\{\lambda + f(U^{n,\sigma}(\lambda, 0) + U^{n,\sigma}(\lambda) \cdot \frac{\partial f}{\partial u}(U^{n,\sigma}(\lambda), 0)\right) \cdot \right\} & L_1^{-1}\left\{U^{n,\sigma}(\lambda) \cdot \frac{\partial f}{\partial v}(U^{n,\sigma}(\lambda), 0) \cdot \right\} \\ 0 & L_2^{-1}\{\mu + g(U^{n,\sigma}(\lambda,))\} \end{pmatrix}$$

and $H(\lambda, \mu, \vec{z}): [C_0^1[a, b]]^2 \to [C_0^1[a, b]]^2$ is completely continuous with

$$\lim_{\|\vec{z}\| \to 0} \left(\frac{\|H(\lambda, \mu, \vec{z})\|}{\|\vec{z}\|}\right) = 0$$

uniformly for (λ, μ) in compact subsets of $\mathbb{R}^2$. Then as in [**3**], (λ, μ, z_1, z_2) solves (4.4) if and only if $(\lambda, \mu, z_1 + U^{n,\sigma}(\lambda), z_2)$ solves (1.1). Moreover, bifurcation from the trivial solution in (4.4) is possible only if $(\lambda, \mu) \in \Sigma^{n,\sigma}$, where

$$\Sigma^{n,\sigma} = \{(\lambda, \mu): \mu_m^{n,\sigma}(\lambda) \text{ for some } \lambda \in \mathbb{R}, m \in Z^+\} \cup \{(\lambda, \mu): \lambda = \lambda_k, k \leqq n\}.$$

From [**4**, Section 2] and (H1), there is a $\bar{\lambda} > \lambda_n$ so that

$$I - L_1^{-1}\left\{\bar{\lambda} + f(U^{n,\sigma}(\bar{\lambda}), 0) + U^{n,\sigma}(\bar{\lambda}) \cdot \frac{\partial f}{\partial u}(U^{n,\sigma}(\bar{\lambda}), 0)\right\}$$

is an invertible operator. It then follows as in [**3**, Section 2] that the Leray–Schauder indices

$$\text{ind}\{I - A(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}) \pm \delta), 0\}$$

are of opposite sigsns for $\delta > 0$ and sufficiently small. In fact, the simplicity of $\mu_m^{n,\sigma}(\bar{\lambda}\Gamma$ as an eigenvalue of (4.3) implies

$$\dim(\ker((I - A(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda})))^2)) = \dim(\ker(I - A(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda})))) = 1. \qquad (4.5)$$

Consequently, $\text{ind}\{I - N(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}) + \delta, \cdot), 0\} \neq \text{ind}\{I - N(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}) - \delta, \cdot), 0\}$ for $\delta > 0$ and sufficiently small. The homotopy invariance of the Leray–Schauder degree and the Alexander–Antman Bifurcation theorem [**1**] imply a transition to a connected set $\mathscr{C}$ of nontrivial solutions to (4.4) which is locally compact and of dimension $\geqq 2$ at every pont as parameter values cross the curve $\mu = \mu_m^{n,\sigma}(\lambda)$, $\lambda > \lambda_n$. Moreover, if $\Gamma(t)$ is any smooth one-dimensional restriction of the parameters (λ, μ) which crosses $\mu = \mu_m^{n,\sigma}(\lambda)$, $\lambda > \lambda_n$ at say $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}))$ and which satisfies $|\Gamma(t)| \to \infty$ as $t \to \pm\infty$, then the restriction of $\mathscr{C}$ to Γ is either unbounded or meets the trivial solutions to (4.4) at a parameter value other than $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}))$.

In fact, much more can be determined. Suppose that $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}))$ satisfies (4.5) and that $\Gamma \cap \Sigma^{n,}$ is a discrete set. Then [**15**, Lemma 1.2] and [**11**, Theorem 2] may be adapted to the situation of (4.4). Consequently, as solutions $(\lambda, \mu_1, z_1, z_2)$ emerge from $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0)$ along Γ, z_1 and z_2 must be such that $z_1 + U^{n,\sigma}(\lambda)$ has $n-1$ simple zeros in (a, b) and z_2 has $m-1$ simple zeros in (a, b). The sign of $(z_1 + U^{n,\sigma}(\lambda))'(a)$ is σ, whereas $z_2'(a)$ can be positive or negative. Moreover, the component $\mathscr{A}$ of the closure of nontrivial solutions to (4.4) (for parameter values along Γ) emanating from $(\lambda, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0)$ can be expressed as $\mathscr{A}^+ \cup \mathscr{A}^-$, with $\mathscr{A}^+$ and $\mathscr{A}^-$ subcontinua of $\mathscr{A}$ so that the nontrivial solutions (λ, μ, z_1, z_2) to (4.4) in $\mathscr{A}^+(\mathscr{A}^-$, respectively) near $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0)$ have the property that $z_1 + U^{n,\sigma}(\lambda)$ has $n-1$ simple zeros in (a, b), $\operatorname{sgn}((z_1 + U^{n,\sigma}(\lambda))'(a)) = \sigma$, z_2 has $m-1$ simple zeros in (a, b) and $\operatorname{sgn} z_2'(a) = +$ ($-$, respectively). Moreover, for a sufficiently small neighbourhood $B((\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0); \delta)$ of $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0)$, $\mathscr{A}^+ \cap \mathscr{A}^- \cup B((\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0); \delta) = \{(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0\}$ and either $\mathscr{A}^+$ and $\mathscr{A}^-$ are both unbounded or $\mathscr{A}^+$ and $\mathscr{A}^-$ meet outside $B((\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0); \delta)$.

Assume now that (H5) and (H6) hold in addition to (H1) and (H2). For Γ, require additionally that Γ cross the cruves $\{\mu = \mu_m^{n,\sigma}(\lambda) \colon \lambda \geqq \lambda_n\}$ and $\{\mu = \mu_m\}$ exactly once at $\lambda = \bar{\lambda}$ and $\lambda = \lambda^* > \lambda_n$, respectively, and that as $t \to \pm\infty$, $(\lambda, \mu) = \Gamma(t)$ implies that $\lambda + \mu < 0$. It follows that $\mathscr{A}^+$ and $\mathscr{A}^-$ exit the nodal sets in which they lie near $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0)$. Otherwise $\mathscr{A}^+$ and $\mathscr{A}^-$ do not intersect and hence are both unbounded. Theorem 3.2 implies that $\mathscr{A}^+$ and $\mathscr{A}^-$ can become only if the parameter values along Γ become unbounded. Since $\lambda + \mu < 0$ if $(\lambda, \mu) = \Gamma(t)$ for $|t|$ sufficiently large, there are no solutions (λ, μ, z_1, z_2) to (4.4) with both $z_1 + U^{n,\sigma}(\lambda)$ and z_2 nonzero as parameter values along Γ become unbounded. This contradiction shows that $\mathscr{A}^+$ and $\mathscr{A}^-$ do exit the nodal sets in which they emanate from $(\bar{\lambda}, \mu_m^{n,\sigma}(\bar{\lambda}), 0, 0)$. Consequently, there are $\mu^+, \mu^- > \mu_m$ so that $(\lambda_n^{m,+}(\mu^+), \mu^+, -U^{n,\sigma}(\lambda_n^{m,+}(\mu^+)), V^{m,+}(\mu^+)) \in \mathscr{A}^+$ and $(\lambda_n^{m,-}(\mu^-), \mu^-, -U^{n,\sigma}(\lambda_n^{m,-}(\mu^-)), V^{m,-}(\mu^-)) \in \mathscr{A}^-$, where for $\mu > \mu_m$, $\lambda_n^{m,\sigma}(\mu)$ is the unique eigenvalue of

$$
\begin{aligned}
L_1 w - f(0, V^{m,\sigma}(\mu)) w &= \lambda w \quad \text{in } (a, b), \\
w(a) = 0 &= w(b),
\end{aligned}
\tag{4.6}
$$

admitting an eigenfunction with $n-1$ simple zeros in (a, b). (Since (H5) and (H6) hold, $\lambda_n^{m,+}(\mu^+), \lambda_n^{m,-}(\mu^-) < \lambda_n$ and so $U^{n,\sigma}(\mu^+)) = 0$ and $U^{n,\sigma}(\lambda_n^{m,-}(\mu^-)) = 0$. If (H5) and (H6) are replaced with (H3) and (H4), such is not the case.) It is now easy to observe that $(\lambda^*, \mu_m, -U^{n,\sigma}(\lambda^*), 0) \in \mathscr{A}^+ \cap \mathscr{A}^-$ and since $\lambda^* > \lambda_n$, $(-U^{n,\sigma}(\lambda^*), 0) \neq (0, 0)$. Hence the portions of $\mathscr{A}^+$ and $\mathscr{A}^-$ which are contained in the indicated nodal sets are in the same component of the nontrivial solutions to (4.4). A similar analysis can be made regarding the transition from $\mathscr{C}_{0,m,\tau}$ to $\mathscr{C}_{n,m,\pm,\tau}$ as parameter values cross the curve $\lambda = \lambda_n^{m,\tau}(\mu)$, $\mu > \mu_m$, where $\lambda_n^{m,\tau}(\mu)$ is given by (4.6). The main result of this article can now be given.

THEOREM 4.1. *Consider* (1.1) *and assume that* (H1)–(H2) *and one of the following sets of additional conditions hold*:

(i) (H3)–(H4),
(ii) (H5)–(H6),
(iii) (H7)–(H10).

Then there is a connected set $\mathscr{C}_{n,m,\sigma,\tau} \subseteq \mathbb{R}^2 \times [C_0^1[a, b]]^2$ *os solutions to* (1.1) *such that*

(a) $\mathscr{C}_{n,m,\sigma,\tau}$ *has dimension* $\geqq 2$ *at every point*;
(b) $\mathscr{C}_{n,m,\sigma,\tau}$ *is locally compact*;
(c) $(\lambda, \mu, u, v) \in \mathscr{C}_{n,m,\sigma,\tau}$ *implies that* u *has* $n-1$ *simple zeros in* (a, b), v *has* $m-1$ *simple zeros in* (a, b), $\operatorname{sgn} u'(a) = \sigma$ *and* $\operatorname{sgn} v'(a) = \tau$;

(d) *$\mathscr{C}_{n,m,\sigma,\tau}$ links $\mathscr{C}_{n,0,\sigma}$ to $\mathscr{C}_{0,m,\tau}$, where $\mathscr{C}_{n,0,\sigma}$ and $\mathscr{C}_{0,m,\tau}$ are given by (4.1) and (4.2), respectively. In particular, the projection of $\mathscr{C}_{n,m,\sigma,\tau}$ includes all points (λ, μ) contained in the region bounded by the curves $\mu = \mu_m^{n,\sigma}(\lambda)$, $\lambda \geqq \lambda_n$ and $\lambda = \lambda_n^{m,\tau}(\mu)$, $\mu \geqq \mu_m$, where $\mu_m^{n,\sigma}(\lambda)$ and $\lambda_n^{m,\tau}(\mu)$ are given by (4.3) and (4.6), respectively.*

Proof. In the case where (ii) is assumed, the theorem follows from combining the observations preceding the theorem with [**1**, Corollary 2.47]. If (i) or (iii) is assumed in place of (ii), slight modifications of the argument are needed, and these are omitted.

REMARKS. (i) If (H2) is replaced with (H2)′ and $\mathbb{R}^2 \times [C_0^1[a, b]]^2$ is replaced with $\{\lambda \in \mathbb{R}: \lambda < \lambda_n + \delta_n\} \times [C_0^1[a, b]]^2$, then a result similar to Theorem 4.1 follows by much the same analysis, providing a link between $\mathscr{C}_{n,0,\sigma}$ and $\mathscr{C}_{0,m,\tau}$ via branches of solutions (λ, μ, u, v) to (1.1) where u has $n-1$ simple zeros in (a, b), v has $m-1$ simple zeros in (a, b), sgn $u'(a) = \sigma$ and sgn $v'(a) = \tau$.

(ii) It should be noted that in the proof of Theorem 4.1 in none of the three cases is the curve Γ identically equal to the line $\lambda = \bar{\lambda}$. In the "competitive" case (i), with the additional assumption $L_1 = L_2$, it is possible to place conditions on f and g so that the bifurcation theoretic arguments of this section may be used to see that $\mathscr{C}_{n,0,\sigma}$ is linked to $\mathscr{C}_{0,m,\tau}$ via $\mathscr{C}_{n,m,\sigma,\tau}$ if the parameters (λ, μ) are restricted to the line $\lambda = \bar{\lambda} > \lambda_n$ if $m \leqq n$ and that $\mathscr{C}_{n,0,\sigma}$ is linked to $\mathscr{C}_{0,m,\tau}$ via $\mathscr{C}_{n,m,\sigma,\tau}$ if the parameters (λ, μ) are restricted to the line $\mu = \bar{\mu} > \mu_m$ if $n \leqq m$. See [**4**, Section 3 and 4]. The "cooperative" and "predatory" cases are qualitatively different. In particular, note that in the "cooperative" case $\mu_m^{n,\sigma}(\lambda) < \mu_m$ for all $\lambda > \lambda_n$ and $\lambda_n^{m,\tau}(\mu) < \lambda_n$ for all $\mu > \mu_m$. Hence, for example, if $\bar{\lambda} > \lambda_n$ is fixed and (λ, μ) are restricted to lie along the line $\lambda = \bar{\lambda}$, then $\mathscr{C}_{n,m,\sigma,\tau}$ does not link to $\mathscr{C}_{0,m,\tau}$. $\mathscr{C}_{n,m,\sigma,\tau}$ must be unbounded for parameter values lying along this line, and the *a priori* estimates of Section 3 imply that if $\mu > \mu_m^{n,\sigma}(\lambda)$, then there is $(u, v) \in [C_0^1[a, b]]^2$ so that $(\bar{\lambda}, \mu, u, v) \in C_{n,m,\sigma,\tau}$. A similar statement is true if $\bar{\mu} > \mu_m$ is fixed. These observations provide an alternative way to see that in the "cooperative" case (ii), the set $\{(\lambda, \mu): \lambda > \lambda_n, \mu > \mu_m\}$ is contained in the projection of $\mathscr{C}_{n,m,\sigma,\tau}$ into $\mathbb{R}^2$.

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